PH-202 **Solution Set # 4** April, 12, 2011

Answer 1: A deck of Cards

(a)As we have 52 cards in total and we are choosing any five from them without any order, so the number of different hands that can be dealt is

$$\text{different Hands} = {}^{52}C_5.$$

(b)There are 13 total cards of a suit. If you choose five cards from the same suit without any order then it's given by ${}^{52}C_5$ and as there are four suits in a deck $4 \times {}^{52}C_5$. Probability of an event is given by number of favourable events ÷ total number of events.

$$\text{Probability of being dealt a flush} = \frac{4 \times {}^{13}C_5}{{}^{52}C_5}$$

(c) There are four colors of one type of a card so if we want to choose two cards of the same type from four , its given by ${}^{4}C_2$ and as there are thirteen types of cards and we could have a pair of any type then ${}^{13}C_1 \times {}^{4}C_2$. The rest three cards could be chosen from any of the twelve types left so we also include ${}^{12}C_3$ and as these three cards could be of any four colors so we multiply $4 \times 4 \times 4$ with the expression also.

$$\text{Probability of being dealt one pair} = P_1 = \frac{13 \times {}^{4}C_2 \times {}^{12}C_3 \times 4^3}{{}^{52}C_5}$$

(d)

$$\text{Probability of not getting one pair after n`th hand} = (1 - P_1)^n$$

Answer 2: The Russian Roulette

(a)

$$\text{Probability of being alive after playing N times} = \left(\frac{5}{6}\right)^N$$

(b)

$$\text{Probability of surviving } (N-1) \text{ turns and then shot} = \left(\frac{5}{6}\right)^{N-1}\left(\frac{1}{6}\right)$$

(c)

$$\begin{aligned}\text{mean number of pulling the trigger} = \langle N \rangle &= \sum_{N=1}^{\infty} N \left(\frac{5}{6}\right)^{N-1}\left(\frac{1}{6}\right) \\ &= \left(\frac{1}{6}\right) \sum_{N=1}^{\infty} N \left(\frac{5}{6}\right)^{N-1}\end{aligned}$$

Using the expression,

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

simplifying it,

$$\sum_{n=1}^{\infty} n\, x^{n-1} = \frac{1}{(1-x)^2}$$

$$\begin{aligned}
\Rightarrow \langle N \rangle &= \left(\frac{1}{6}\right) \cdot \frac{1}{(1-\frac{5}{6})^2} \\
&= \left(\frac{1}{6}\right) \cdot 36 \\
\langle N \rangle &= 6
\end{aligned}$$

(d) Probability that my friend is taking the first shot and is dying in the first shot is $\frac{1}{6}$, probability of him dying on the second shot is $\frac{5}{6}\frac{5}{6}\frac{1}{6}$ similarly on the third shot $\frac{5}{6}\frac{5}{6}\frac{5}{6}\frac{5}{6}\frac{1}{6}$ and so on.

$$\begin{aligned}
\text{the probability that your friend dies if he takes the first shot} &= \left(\frac{1}{6} + \frac{5}{6}\frac{5}{6}\frac{1}{6} + (\frac{5}{6})^4 \frac{1}{6} + ...\right) \\
&= \frac{1}{6}\left(1 + \frac{25}{36} + ...\right) \\
\text{This becomes a geometric series, so we use geometric series sum formula} & \\
&= \frac{1}{6}\left(\frac{1}{1-\frac{25}{36}}\right) \\
&= \frac{1}{6}\left(\frac{36}{11}\right) \\
&= \frac{6}{11}
\end{aligned}$$

Probability that my friend dies when I take the first shot is zero for the first shot, for the second shot it is $\frac{5}{6}\frac{1}{6}$ for the third shot it is $\frac{5}{6}\frac{1}{6}\frac{5}{6}\frac{5}{6}$ and so on.

$$\begin{aligned}
\text{the probability that my friend dies if I take the first shot} &= \left(\frac{1}{6}\frac{5}{6} + (\frac{5}{6})^3 \frac{1}{6} + ..\right) \\
\text{again using the geometric series sum formula} & \\
&= \frac{5}{36}\left(\frac{1}{1-\frac{25}{36}}\right) \\
&= \frac{5}{11}
\end{aligned}$$

Answer 3: A Random Walk.

(a) If N is even, then we use,

$$P = \frac{N!}{n!(N-n)!} p^n q^{(N-n)}, \qquad (1)$$

Where p is the probability to move left, q is the probability to move right, n is the number of steps towards left and $(N-n)$ is the number of steps towards right. Since according to the statement of problem3 $p = q$ and $p + q = 1$, $\Rightarrow p = q = 1/2$.
Thus equation (1) will become,

$$\begin{aligned}
P(n, N) &= \frac{N!}{n!(N-n)!} \left(\frac{1}{2}\right)^n \left(\frac{1}{2}\right)^{N-n}, \\
&= \frac{N!}{n!(N-n)!} \left(\frac{1}{2}\right)^N,
\end{aligned}$$

Since $2n = N$ or $n = N/2$, thus probability will be,

$$\begin{aligned} P(n,N) &= \frac{N!}{(\frac{N}{2})!(N-\frac{N}{2})!}\left(\frac{1}{2}\right)^N, \\ &= \frac{N!}{[(\frac{N}{2})!]^2}\left(\frac{1}{2}\right)^N. \end{aligned}$$

(b) If N is odd, then $P = 0$.

Answer 4: More Random Walk

Now if the drunk is taking two steps to the right and one to the left then his probability of going right is $\frac{2}{3}$ and to the left it is $\frac{1}{3}$. Then if he is taking N total steps then

$$\begin{aligned} \text{steps to the left} &= \frac{N}{3} \\ \text{Steps to the right} &= \frac{2N}{3} \\ \text{Net displacement to the right} &= \frac{2N}{3} - \frac{N}{3} \\ &= \frac{N}{3} \end{aligned}$$

Answer 5: Gas in a box

(a)

$$\begin{aligned} P &= \sum_{N=0}^{N_o} {}^{N_o}C_N \left(\frac{V}{V_o}\right)^N \left(1-\frac{V}{V_o}\right)^{N_o-N} \\ \langle N \rangle &= \sum_{N=0}^{N_o} {}^{N_o}C_N N \left(\frac{V}{V_o}\right)^N \left(1-\frac{V}{V_o}\right)^{N_o-N} \end{aligned}$$

Now if we use the following formula

$$\begin{aligned} (x+y)^N &= \sum_{N=0}^{N_o} {}^{N_o}C_N x^N y^{N_o-N} \\ N(x+y)^{N-1} &= \sum_{N=0}^{N_o} {}^{N_o}C_N N x^{N-1} y^{N_o-N} \end{aligned}$$

Putting $x = \frac{V}{V_o}$ and $y = 1 - \frac{V}{V_o}$

$$\begin{aligned} N\left(\frac{V}{V_o} + 1 - \frac{V}{V_o}\right)^{N-1} &= \sum_{N=0}^{N_o} {}^{N_o}C_N N \left(\frac{V}{V_o}\right)^{N-1} \left(1-\frac{V}{V_o}\right)^{N_o-N} \\ N(1)^{N-1} &= \sum_{N=0}^{N_o} {}^{N_o}C_N N \left(\frac{V}{V_o}\right)^{N} \left(1-\frac{V}{V_o}\right)^{N_o-N} \frac{V_o}{V} \\ \frac{VN}{V_o} &= \sum_{N=0}^{N_o} {}^{N_o}C_N N \left(\frac{V}{V_o}\right)^{N} \left(1-\frac{V}{V_o}\right)^{N_o-N} \end{aligned}$$

here this equals to the value of $\langle N \rangle$, so we get

$$N\frac{V}{V_o} = \langle N \rangle.$$

(b)

$$\langle N^2 \rangle = \sum_{N=0}^{N_o} {}^{N_o}C_N N^2 \left(\frac{V}{V_o}\right)^N \left(1 - \frac{V}{V_o}\right)^{N_o - N}$$

again using

$$(x+y)^N = \sum_{N=0}^{N_o} {}^{N_o}C_N x^N y^{N_o - N}$$

$$N(x+y)^{N-1} = \sum_{N=0}^{N_o} {}^{N_o}C_N N x^{N-1} y^{N_o - N}$$

$$N(N-1)(x+y)^{N-2} = \sum_{N=0}^{N_o} {}^{N_o}C_N N(N-1) x^{N-2} y^{N_o - N}$$

$$= \sum_{N=0}^{N_o} {}^{N_o}C_N (N^2 - N) x^{N-2} y^{N_o - N}$$

$$= \sum_{N=0}^{N_o} {}^{N_o}C_N N^2 x^{N-2} y^{N_o - N} - \sum_{N=0}^{N_o} {}^{N_o}C_N N x^{N-2} y^{N_o - N}$$

Putting $x = \frac{V}{V_o}$ and $y = 1 - \frac{V}{V_o}$

$$N(N-1)\left(\frac{V}{V_o} + 1 - \frac{V}{V_o}\right)^{N-2} = \sum_{N=0}^{N_o} {}^{N_o}C_N N^2 \left(\frac{V}{V_o}\right)^{N-2} \left(1 - \frac{V}{V_o}\right)^{N_o - N}$$

$$- \sum_{N=0}^{N_o} {}^{N_o}C_N N \left(\frac{V}{V_o}\right)^{N-2} \left(1 - \frac{V}{V_o}\right)^{N_o - N}$$

$$N(N-1) = \frac{{V_o}^2}{V^2} \sum_{N=0}^{N_o} {}^{N_o}C_N N^2 \left(\frac{V}{V_o}\right)^{N} \left(1 - \frac{V}{V_o}\right)^{N_o - N}$$

$$- \frac{{V_o}^2}{V^2} \sum_{N=0}^{N_o} {}^{N_o}C_N N \left(\frac{V}{V_o}\right)^{N} \left(1 - \frac{V}{V_o}\right)^{N_o - N}$$

Now above expressions are equal to $\langle N \rangle$ and $\langle N^2 \rangle$

$$
\begin{aligned}
N(N-1) &= \frac{{V_o}^2}{V^2}\left\{\langle N^2\rangle - \langle N\rangle\right\} \\
\frac{V^2}{{V_o}^2}N(N-1) &= \left\{\langle N^2\rangle - \langle N\rangle\right\} \\
\langle N^2\rangle &= \langle N\rangle + \frac{V^2}{{V_o}^2}N(N-1) \\
&= N\frac{V}{V_o} + \frac{V^2}{{V_o}^2}N(N-1) \\
&= N\frac{V}{V_o} + \frac{V^2}{{V_o}^2}N^2 - \frac{V^2}{{V_o}^2}N \\
&= N\left(\frac{V}{V_o} - \frac{V^2}{{V_o}^2}\right) + \frac{V^2}{{V_o}^2}N^2 \\
&= N\left(\frac{VV_o - V^2}{{V_o}^2}\right) + \frac{V^2}{{V_o}^2}N^2 \\
&= N\frac{V}{V_o}\left\{1 - \frac{V}{V_o} + N\frac{V}{V_o}\right\}.
\end{aligned}
$$

To calculate relative dispersion, we make use of the following formula

$$
\begin{aligned}
\frac{\left\langle (N-\langle N\rangle)^2\right\rangle}{\langle N\rangle^2} &= \frac{\langle N^2\rangle - \langle N\rangle^2}{\langle N\rangle^2} \\
&= \frac{\langle N^2\rangle}{\langle N\rangle^2} - 1, \\
&= \frac{N\frac{V}{V_o}(1-\frac{V}{V_o}+N\frac{V}{V_o})}{(N\frac{V}{V_o})(N\frac{V}{V_o})} - 1 \\
&= \frac{1-\frac{V}{V_o}+N\frac{V}{V_o}-N\frac{V}{V_o}}{N\frac{V}{V_o}}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1-\frac{V}{V_o}}{N\frac{V}{V_o}} \\
&= \frac{1}{N\frac{V}{V_o}} - \frac{\frac{V}{V_o}}{N\frac{V}{V_o}} \\
&= \frac{V_o}{NV} - \frac{1}{N} \\
&= \frac{1}{N}(\frac{V_o}{V} - 1)
\end{aligned}
$$

(c) $V << 0$,

$$
\frac{\left\langle (N-\langle N\rangle)^2\right\rangle}{\langle N\rangle^2} = \infty,
$$

(d)$V \to V_o$,

$$
\frac{\left\langle (N-\langle N\rangle)^2\right\rangle}{\langle N\rangle^2} = 0.
$$

Answer 6: Angular Momentum

$$\begin{aligned} P_h + P_o + P_{-\hbar} &= 0, \\ P_\hbar \hbar + P_o(o) + P_{-\hbar}(-\hbar) &= \frac{1}{3}\hbar, \\ P_\hbar \hbar^2 + P_o(o)^2 + P_{-\hbar}(-\hbar)^2 &= \frac{2}{3}\hbar^2, \end{aligned}$$

$$\begin{aligned} \hbar(P_\hbar - P_{-\hbar}) &= \frac{1}{3}\hbar, \\ \Rightarrow P_\hbar - P_{-\hbar} &= \frac{1}{3}, \qquad (2) \end{aligned}$$

$$\begin{aligned} \hbar^2(P_\hbar + P_{-\hbar}) &= \frac{2}{3}\hbar^2, \\ \Rightarrow P_\hbar + P_{-\hbar} &= \frac{2}{3}, \qquad (3) \\ \frac{2}{3} + P_o &= 0, \\ \Rightarrow P_0 &= -\frac{2}{3}, \end{aligned}$$

By adding equation (2) and equation (3), we get,

$$\begin{aligned} 2P_\hbar &= 1, \\ \Rightarrow P_\hbar &= \frac{1}{2}. \\ \frac{1}{2} - P_{-\hbar} &= \frac{1}{3}, \\ \frac{1}{2} - \frac{1}{3} &= P_{-\hbar}, \\ P_{-\hbar} &= \frac{1}{6}. \end{aligned}$$

Answer 7: Electron Energy

(a) For $E > 0$,

$$\begin{aligned} P &= \int_0^\infty P(E)dE, \\ &= \int_0^\infty (0.8)\cdot\frac{1}{b}\cdot e^{-\frac{E}{b}}dE \\ &= \left(\frac{0.8}{b}\right)\int_0^\infty e^{-\frac{E}{b}}dE \\ &= \left(\frac{0.8}{b}\right)\int_0^\infty e^{-(+\frac{1}{b})E}dE \end{aligned}$$

Since,

$$\begin{aligned} \int_0^\infty xe^{-ax}dx &= \frac{1}{a}, \\ \Rightarrow P &= \left(\frac{0.8}{b}\right)\cdot\frac{1}{1/b}, \\ &= \left(\frac{0.8}{b}\right)\cdot b, \\ &= 0.8. \end{aligned}$$

(b) Mean energy of electron will be as follows,

$$\langle E\rangle = \int_{-\infty}^{0} 0.2E\delta(E+E_o)dE + \int_{0}^{\infty}\left(\frac{0.8}{b}\right)Ee^{-\frac{E}{b}}dE$$

Let second integral is equal to I, then,

$$\begin{aligned} I &= \int_{0}^{\infty}\left(\frac{0.8}{b}\right)Ee^{-\frac{E}{b}}dE, \\ &= \left(\frac{0.8}{b}\right)\int_{0}^{\infty} Ee^{-\frac{E}{b}}dE, \end{aligned}$$

Let, $x=\frac{E}{b}, \Rightarrow E = xb$ and $dE = bdx$.

$$\begin{aligned} I &= \left(\frac{0.8}{b}\right)\int_{0}^{\infty}(xb)e^{-x}bdx, \\ &= 0.8b\int_{0}^{\infty} xe^{-x}dx, \\ &= 0.8b\left| x\cdot\frac{e^{-x}}{-1}\right|_{0}^{\infty} - 0.8b\int_{0}^{\infty}\frac{e^{-x}}{-1}dx, \\ &= 0.8b\left| x\cdot\frac{e^{-x}}{-1}\right|_{0}^{\infty} + 0.8b\left|\frac{e^{-x}}{-1}\right|_{0}^{\infty}, \\ &= 0.8b(0) + 0.8b(1), \\ &= 0.8b. \end{aligned}$$

Thus mean energy of electron will be,

$$\langle E\rangle = \int_{-\infty}^{0} 0.2E\delta(E+E_o)dE + 0.8b.$$

Answer 8: The Photons

$$\begin{aligned} \langle n\rangle &= \sum_{n=0}^{\infty} n(1-a)a^n, \\ &= (1-a)\sum_{n=0}^{\infty} na^n\cdot, \end{aligned} \quad (4)$$

Since,

$$\begin{aligned} \sum_{n=0}^{\infty} x^n &= \frac{1}{1-x}, \\ \Rightarrow \sum_{n=0}^{\infty} a^n &= \frac{1}{1-a}, \\ \sum_{n=0}^{\infty} na^{n-1} &= \frac{1}{(1-a)^2}, \\ \sum_{n=0}^{\infty} na^n &= \frac{a}{(1-a)^2}, \end{aligned}$$

Thus equation (4)will become,

$$\begin{aligned} \langle n\rangle &= (1-a)\cdot\frac{a}{(1-a)^2}, \\ &= \frac{a}{1-a}. \end{aligned}$$

Now,

$$
\begin{aligned}
\langle n^2 \rangle &= \sum_{n=0}^{\infty} n^2 (1-a) a^n, \\
&= (1-a) \sum_{n=0}^{\infty} n^2 a^n \cdot, \qquad (5)
\end{aligned}
$$

Again,

$$
\begin{aligned}
\sum_{n=0}^{\infty} x^n &= \frac{1}{1-x}, \\
\Rightarrow \sum_{n=0}^{\infty} a^n &= \frac{1}{1-a}, \\
\sum_{n=0}^{\infty} n a^{n-1} &= \frac{1}{(1-a)^2}, \\
\sum_{n=0}^{\infty} n(n-1) a^{n-2} &= \frac{2}{(1-a)^3}, \\
\sum_{n=0}^{\infty} n^2 a^{n-2} - \sum_{n=0}^{\infty} n^2 a^{n-2} &= \frac{2}{(1-a)^3}, \\
\left(\frac{1}{a^2}\right)\left[\sum_{n=0}^{\infty} n^2 a^n - \sum_{n=0}^{\infty} n^2 a^n\right] &= \frac{2}{(1-a)^3}, \\
\left(\frac{1}{a^2}\right)\left[\langle n^2 \rangle - \langle n \rangle\right] &= \frac{2}{(1-a)^3}, \\
\left(\frac{1}{a^2}\right)\left[\langle n^2 \rangle - \frac{a}{(1-a)}\right] &= \frac{2}{(1-a)^3}, \\
\left[\langle n^2 \rangle - \frac{a}{(1-a)}\right] &= \frac{2a^2}{(1-a)^3}, \\
\langle n^2 \rangle &= \frac{2a^2}{(1-a)^3} + \frac{a}{(1-a)}, \\
&= \frac{2a^2(1-a)^2 + a}{(1-a)}.
\end{aligned}
$$

Thus equation (5) will become,

$$
\begin{aligned}
\langle n^2 \rangle &= (1-a) \cdot \frac{2a^2(1-a)^2 + a}{(1-a)}, \\
&= 2a^2(1-a)^2 + a, \\
\langle n^2 \rangle - \langle n \rangle^2 &= \{2a^2(1-a)^2 + a\} - \frac{a^2}{(1-a)^2}, \\
&= \frac{2a^2(1-a)^4 + a(1-a)^2 - a^2}{(1-a)^2}.
\end{aligned}
$$

<u>**Answer 9**</u>**: 2-Dimensional Gas**,

(a) The phase space volume $\Omega(E)$ of allowed states is,

$$\Omega(E) = \frac{d\omega(E)}{dE} \times \Delta,$$

where $\omega(E)$ is volume of solid region bounded by energy surface E.
$\Omega(E)$ is volume in phase space of the region having between E and $E+\Delta$.

$$\Omega(E) = \frac{1}{h_0^{2N}} \int d^{2N}q d^{2N}p,$$

here $2N$ is the degree of freedom.
Now,

$$\omega(E) = \frac{A^N}{h_0^{2N}} \int_{H \leq E} d^{2N}p. \tag{6}$$

While integration over momenta is,

$$\int_{H \leq E} d^{2N}p = \int_{\sqrt{p_N^2} \leq \sqrt{2mE} = R} d^{2N}p.$$

This integral is simply area of $2N$-dimensional sphere with radius $R = \sqrt{2mE}$.
By using relation for volume of d-sphere,

$$V_d = \frac{\pi^{d/2} R^d}{\frac{d}{2}\Gamma(\frac{d}{2})}\cdot$$

Therefore, for $d = 2N$

$$V_d = \frac{\pi^N R^{2N}}{N\Gamma(N)}\cdot$$

By Equation(6),

$$\begin{aligned}
\omega(E) &= \frac{A^N \pi^N R^{2N}}{h_0^{2N} N\Gamma(N)} \\
&= \frac{A^N (2mE\pi)^N}{h_0^{2N} N!} \\
\Omega(E) &= \frac{A^N (2\pi m)^N E^{N-1}\Delta}{h_0^{2N}(N-1)!}
\end{aligned}$$

For the large N, $N-1 \approx N$.

$$\Omega(E) = \frac{A^N (2\pi m)^N E^N \Delta}{h_0^{2N}(N)!}$$

Therefore, entropy S

$$\begin{aligned}
S &= K\ln\Omega(E) \qquad (7)\\
&= k\Big(N\ln A + N\ln(2m\pi) + N\ln E + \ln\Delta - 2N\ln h_0 - \ln N!\Big) \qquad (8)
\end{aligned}$$

Now,

$$\frac{1}{T} = \left(\frac{dS}{dE}\right)_A.$$

By differentiating Equation(8) with respect to E, energy in terms of temperature can be written as

$$\begin{aligned}
\frac{1}{T} &= kN\frac{d\ln E}{dE} \\
&= \frac{kN}{E} \\
E &= NkT.
\end{aligned}$$

(b) Now from the first law of thermodynamics,

$$\begin{aligned}
dE &= dQ - pdA \\
&= TdS - pdA \\
\frac{1}{T} &= \left(\frac{dS}{dE}\right)_A. \\
\frac{p}{T} &= \left(\frac{dS}{dA}\right)_E.
\end{aligned}$$

By differentiating Equation(8) with respect to A, the relation between p, T and A can be written,

$$\begin{aligned}
\frac{p}{T} &= kN\frac{d\ln A}{dA} \\
\frac{p}{T} &= \frac{kN}{A}.
\end{aligned}$$

Answer 10: Balls in large dimensions,

For the ideal gas, $\omega(E)$ is

$$\omega(E) = \frac{V^N(2mE\pi)^{3N/2}}{h^{3N}(\frac{3N}{2})!}. \tag{9}$$

By using relation for entropy,

$$\begin{aligned}
S &= k\ln\omega(E) && (10) \\
&= k\ln\left[\frac{V^N(2mE\pi)^{3N/2}}{h^{3N}(\frac{3N}{2})!}\right] && (11) \\
&= k\left[N\ln V + (\frac{3N}{2})\ln(2\pi mE) - 3N\ln h - \ln(\frac{3N}{2})!\right] && (12)
\end{aligned}$$

From Boltzmann's definition for entropy,

$$\begin{aligned} S &= k\ln\Omega(E) && (13) \\ \Omega(E) &= \frac{d\omega(E)}{dE}\Delta. && (14) \end{aligned}$$

Take derivative of Equation(9) with respect to E, $\Omega(E)$ becomes

$$\begin{aligned} \Omega(E) &= \frac{V^N(\frac{3N}{2})(2\pi mE)^{3N/2-1}\Delta}{h^{3N}(\frac{3N}{2})!} \\ &= \frac{V^N(2\pi mE)^{\frac{3N}{2}-1}\Delta}{h^{3N}(\frac{3N}{2}-1)!}. \end{aligned}$$

From Equation(13),

$$\begin{aligned} S &= k\ln\left(\frac{V^N(2\pi mE)^{\frac{3N}{2}-1}\Delta}{h^{3N}(\frac{3N}{2}-1)!}\right) \\ &= k\left(N\ln V + (\frac{3N}{2}-1)\ln(2\pi mE) + \ln\Delta - 3N\ln h - \ln(\frac{3N}{2}-1)!\right) \\ &= k\left(N\ln V + (\frac{3N}{2})\ln(2\pi mE) - 3N\ln h - \ln(\frac{3N}{2})!\right) \qquad \text{In the limit of large} N, \quad (\frac{3N}{2}-1 \approx \frac{3N}{2}) \\ &= k\ln\omega(E). \end{aligned}$$

Hence proved.

<u>Answer</u> 11 : A System of Classical Harmonic Oscillators

We know that for one harmonic oscillator in one dimension, energy is

$$E = \frac{P^2}{2m} + \frac{mw^2x^2}{2}$$

then for a 3-D harmonic oscillator

$$E = \frac{P_x^2}{2m} + \frac{P_y^2}{2m} + \frac{P_z^2}{2m} + \frac{mw^2(x^2+y^2+z^2)}{2}$$

Notice, that in the above equation we have
i) Three harmonic Oscillators one for each direction.
ii) We have taken same frequency of vibration i.e w for all of them.
Now, if we have N harmonic oscillator

$$E = \sum_i^{3N} \frac{P_i^2}{2m} + \frac{mw^2x_i^2}{2}$$

We want to find the allowed phase space volume if $E = E_0 + \Delta$. For that first we find allowed phase space volume if $E < E_0$.

$$
\begin{aligned}
\omega(E) = \int_{E<E_0} dx^{3N} dp^{3N} & \\
\text{if } E &< E_0 \\
\frac{1}{2m}\sum_{i=0}^{3N}(p_i^2 + m^2 w^2 x_i^2) &< E_0 \qquad \text{; value of E taken from the statement.}
\end{aligned}
$$

To solve the above integral , we change variables,

$$
\begin{aligned}
\text{Let } mwx_i &= x'_i \\
dx_i &= \frac{1}{mw} dx'_i \\
\omega(E) &= \frac{1}{(mw)^{3N}} \int dx'^{3N} dp^{3N}
\end{aligned}
$$

The integral now is just a 6N dimensional sphere with radius $\sqrt{2mE_0}$. So,

$$
\begin{aligned}
\omega(E) &= \frac{1}{(mw)^{3N}} \frac{\pi^{3N}\sqrt{2mE_0}^{6N}}{3N\Gamma(3N)} \\
&= \frac{1}{(mw)^{3N}} \frac{\pi^{3N}\sqrt{2mE_0}^{3N}}{3N!}
\end{aligned}
$$

As we know the number of allowed microstates is given by

$$
\begin{aligned}
\Omega(E) &= \frac{\partial \omega}{\partial E} \\
&= \frac{1}{(mw)^{3N} 3N!} \pi^{3N} \sqrt{2mE_0}^{3N-1} 2m
\end{aligned}
$$

(b) As given in the statement $S = k \log \Omega$. To calculate entropy S we take log of the above equation.

$$
S = k3N \log(\frac{2m\pi}{wm}) + k \log(\frac{1}{3n!}) + k \log E_0 \times (3N-1)
$$

Now if we differentiate S with respect to E we get

$$
\frac{\partial S}{\partial E} = \frac{k(3N-1)}{E_0}
$$

this is equal to inverse of T (which is the definition of thermodynamical temperature)

$$
\begin{aligned}
\frac{k(3N-1)}{E_0} &= \frac{1}{T} \\
E_0 &= (3N-1)kT
\end{aligned}
$$

here as N is very large so we ignore the

$$
\begin{aligned}
E_0 &= 3NkT \\
\text{For calculating C} & \\
\frac{\partial E}{\partial T} &= 3Nk = 3R
\end{aligned}
$$

The heat capacity of ideal gas is half this value i.e $\frac{3}{2}R$ because for ideal gases we consider no attractions among their particles but for the above oscillators we have considered K.E + P.E terms.